Fixed-parameter tractable computation of Reshetikhin–Turaev knot polynomials via tensor networks

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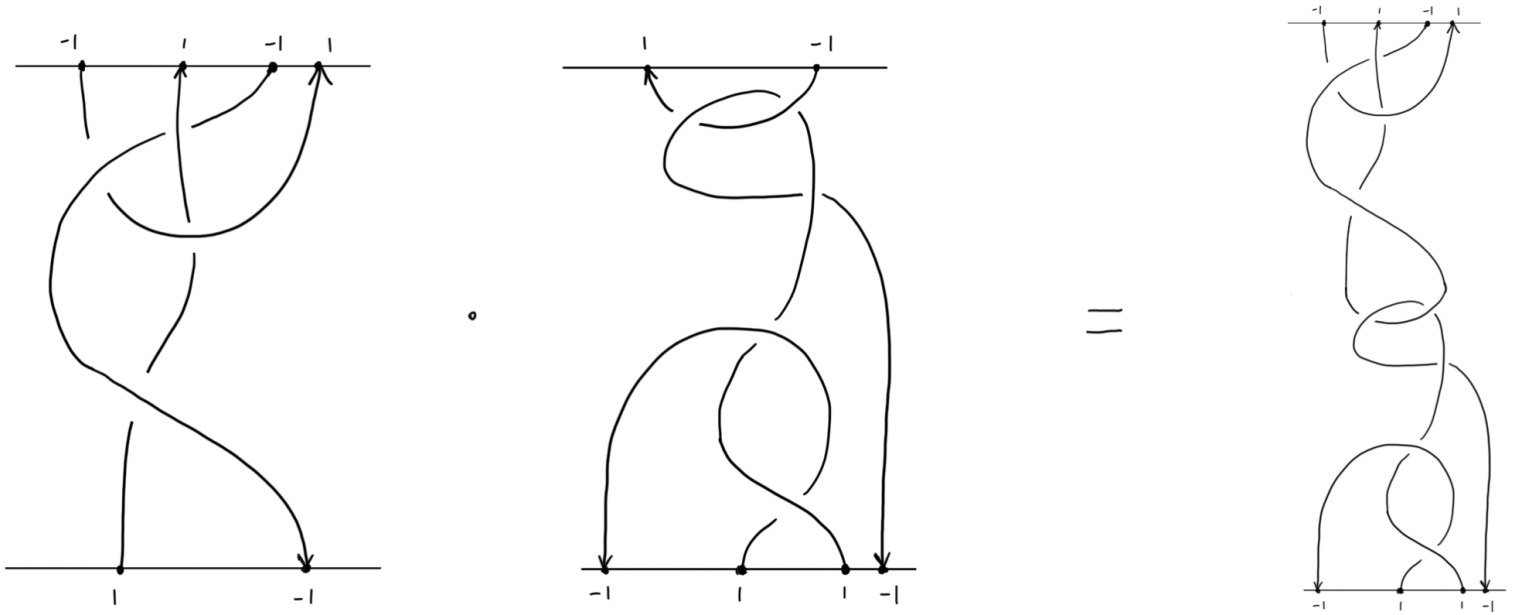

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- A vector space V over a field $\mathbb{F}$.
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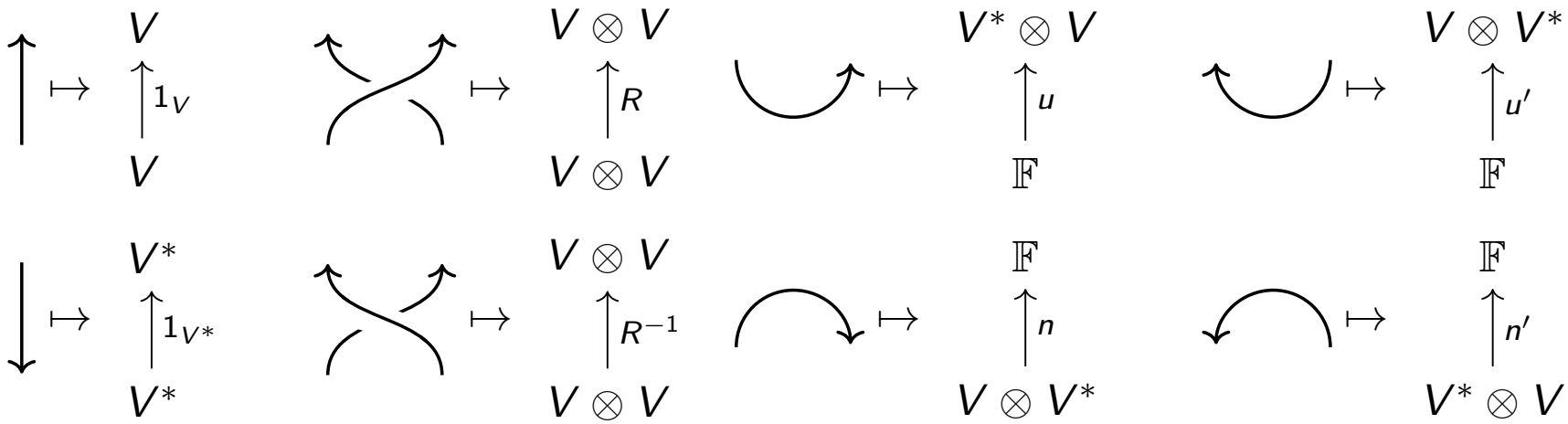
$$(I \otimes R) \circ (R \otimes I) \circ (I \otimes R) = (R \otimes I) \circ (I \otimes R) \circ (R \otimes I).$$

Reshetikhin–Turaev functor: oriented tangles $\mapsto$ vector spaces



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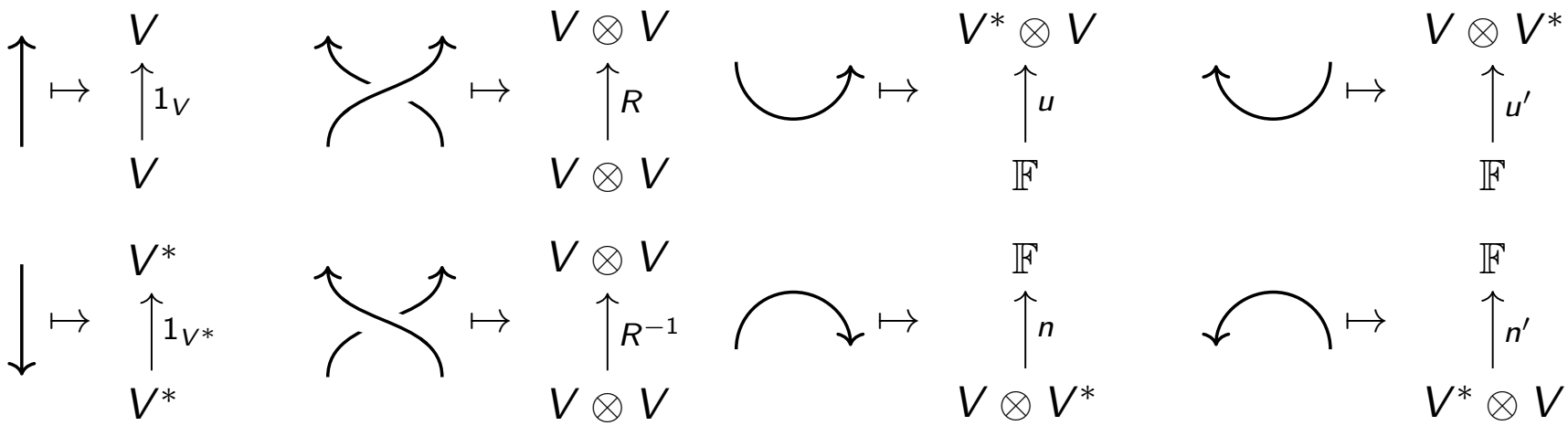
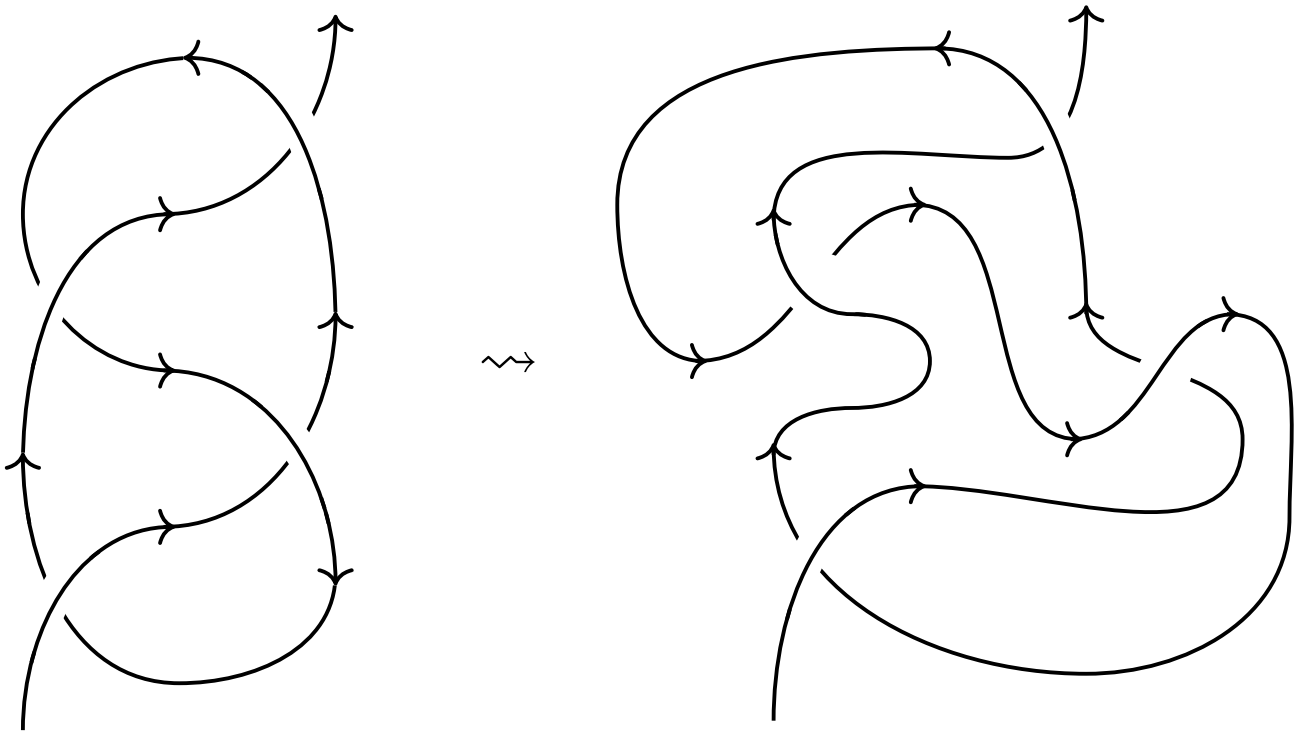


$$u(1) = \sum_i e_i^* \otimes h(e_i), \quad u'(1) = \sum_i e_i \otimes e_i^*,$$

$$n(x \otimes f) = f(h^{-1}(x)), \quad n'(f \otimes x) = f(x)$$

Reshetikhin–Turaev: oriented tangles $\mapsto$ vector spaces

Example: the 4_1 knot



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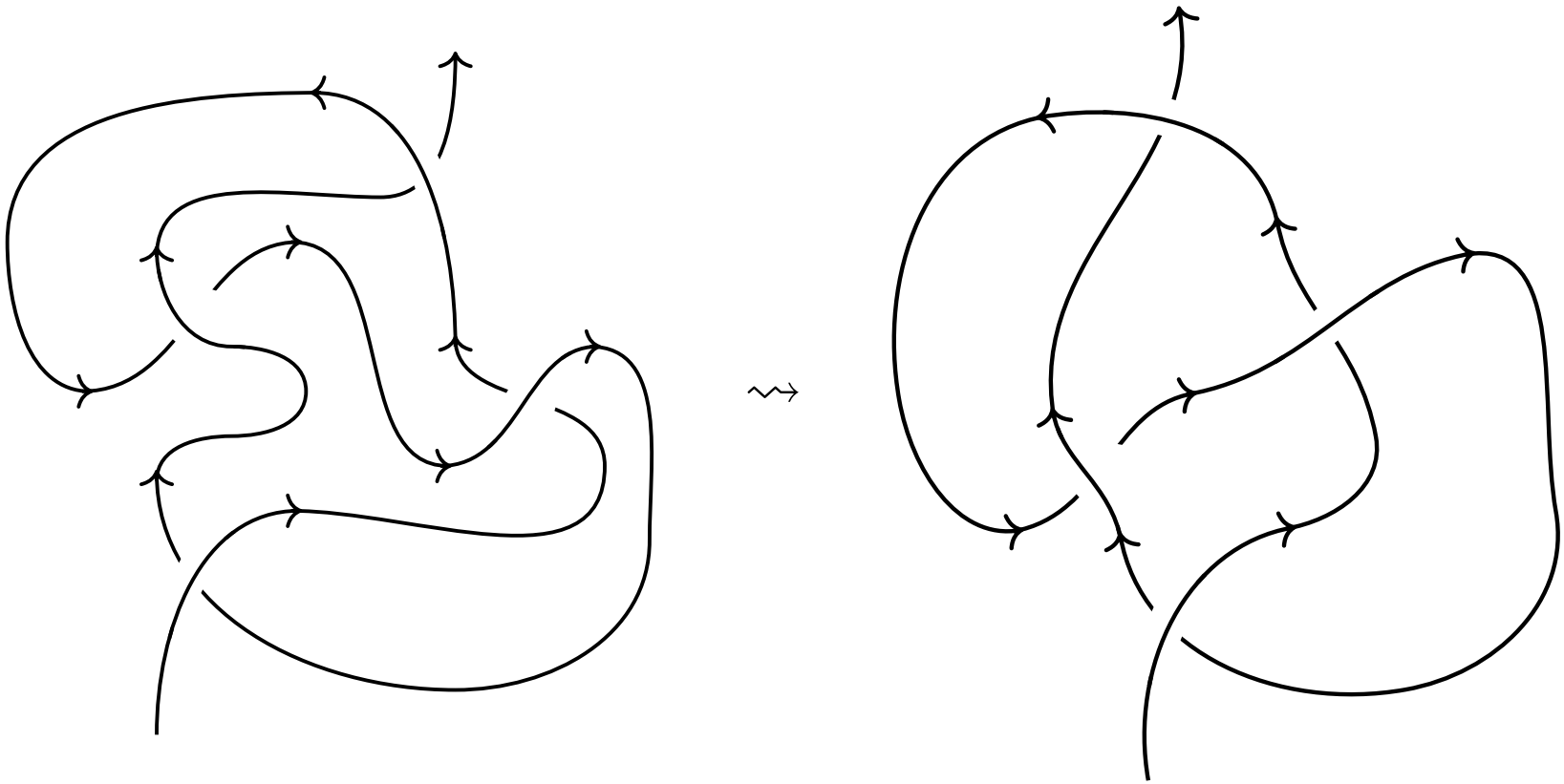

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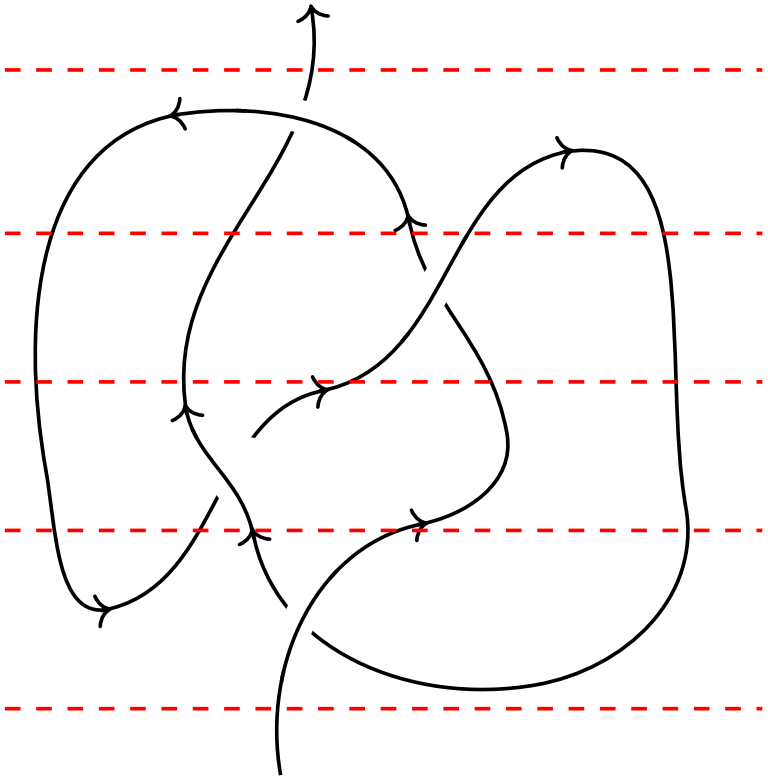

$\mapsto$

$$
\begin{array}{ll}
V & \\
\uparrow & \left(\left(n' \otimes 1_V\right) \circ \left(1_V \otimes R^{-1}\right)\right) \otimes n \\
V^{\otimes 5} & \\
\uparrow & 1_V \otimes 1_V \otimes R \otimes 1_V \\
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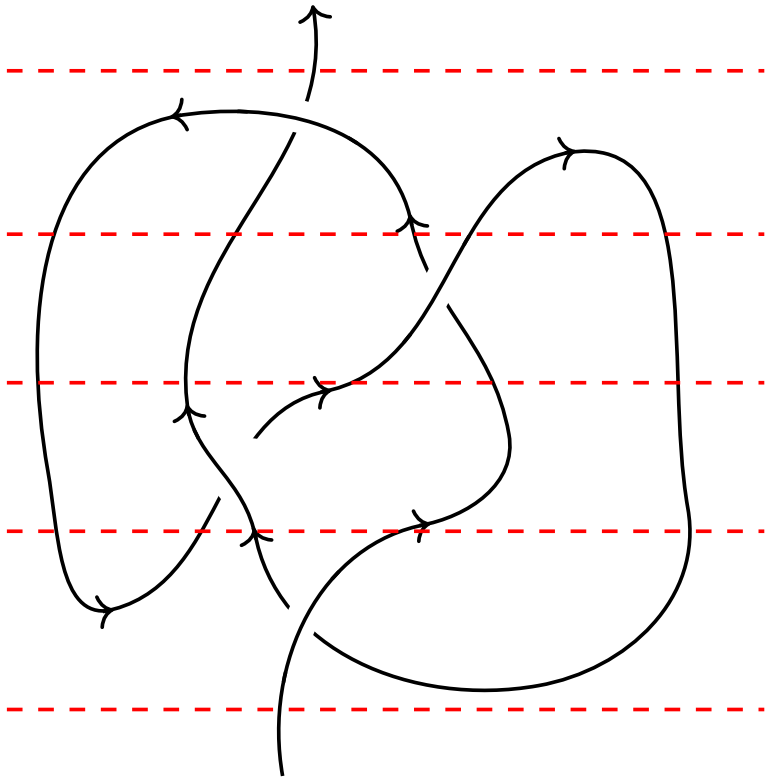

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$$

Fact

The endomorphism on V we obtain is a scalar multiple of 1_V.
Plugging indeterminants in R and h gives polynomial invariants.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ (and h) become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)_{e_i \otimes e_j}^{e_k \otimes e_l}$ (and $h_{e_i}^{e_j}$).

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$$\sum_{\substack{a_1,\cdots,a_{2n-1} \in \mathcal{B} \\ a_0 = a_{2n} = e_1}} \underbrace{\left(R^{\pm 1}\right)_{a_0 \otimes a_1}^{a_2 \otimes a_3} \cdots \left(R^{\pm 1}\right)_{a_{2n-3} \otimes a_{2n-2}}^{a_{2n-1} \otimes a_{2n}}}_{\text{a product of length } n},$$

where n is the number of crossings of the knot.

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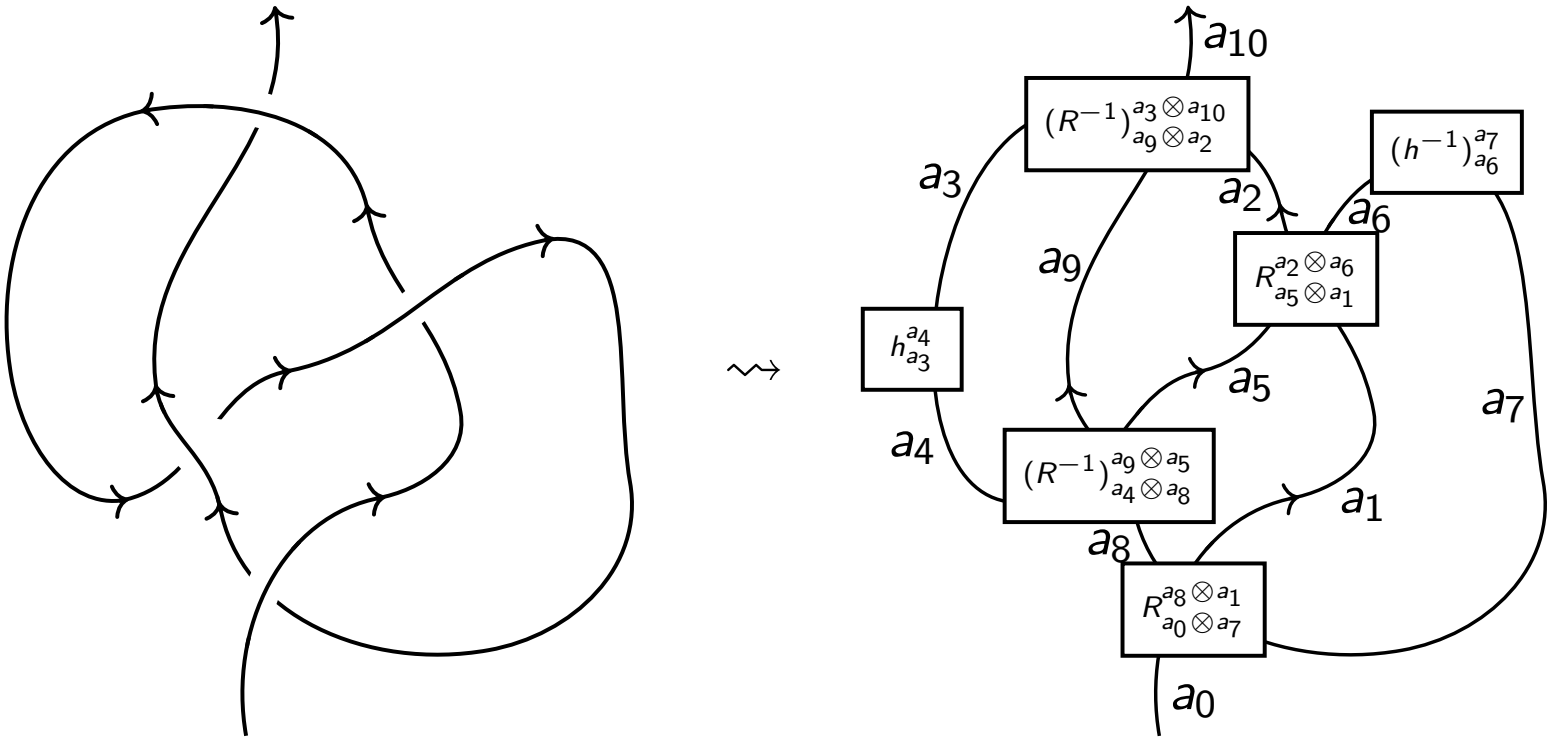
Hence naively it requires

$$n \cdot (\dim V)^{2n-1}$$

multiplications and additions to compute the eigenvalue.

Example: the 4_1 knot again



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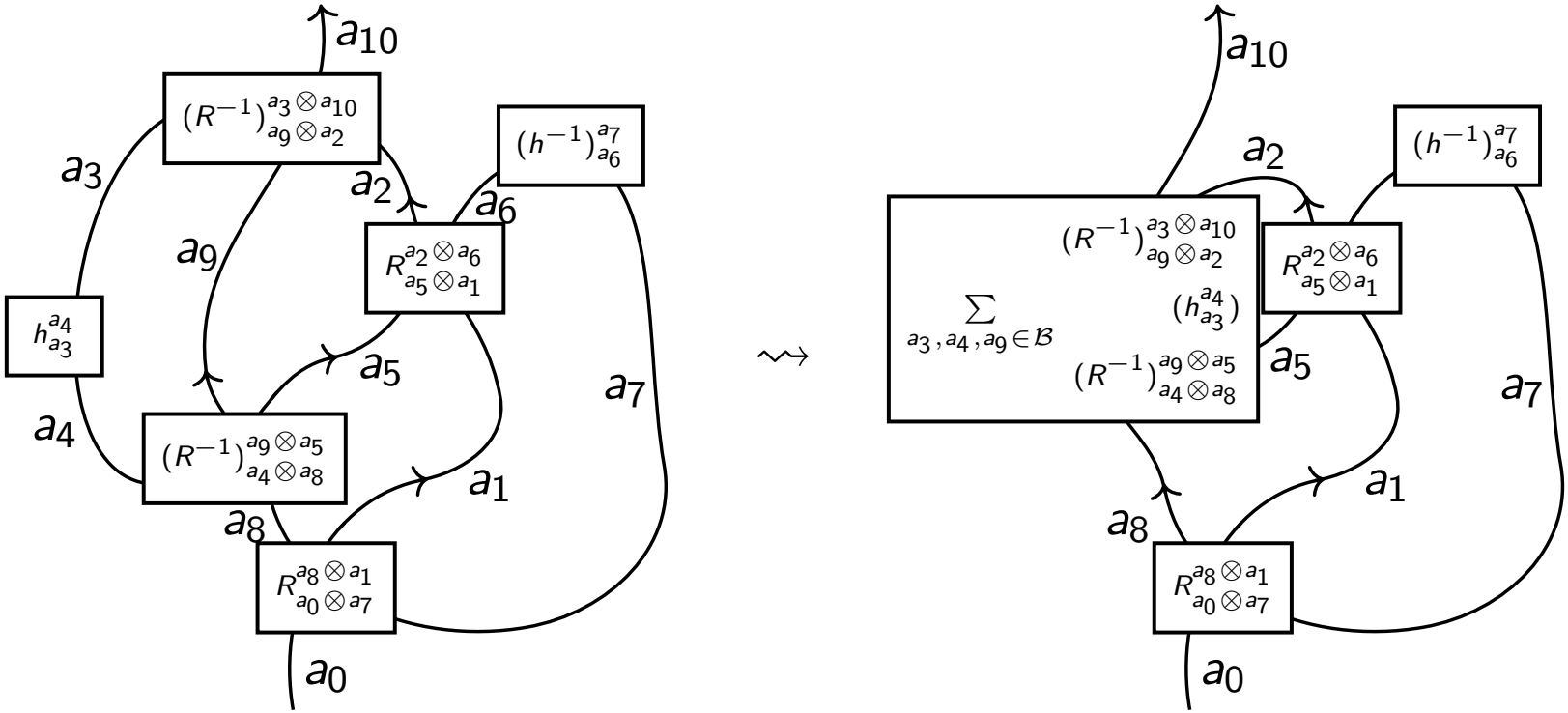

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Definition (Tensor)

An (equi-dimensional) n-tensor T over a ring $\mathcal{R}$ is a tuple $(T_{i_1,\dots,i_n})$ of elements of $\mathcal{R}$ with $i_j = 1, \dots, d$ for each j.

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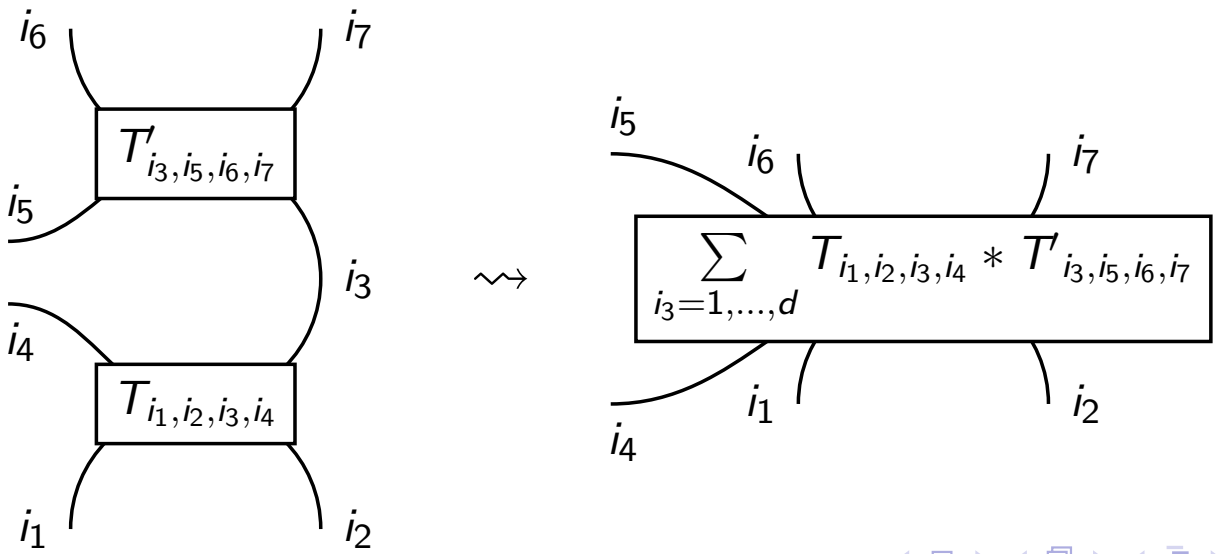

Computing a contraction of c tensors summing along s indices is

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Definition (Contraction-width)

Let $\mathfrak{S}(\Gamma)$ denote the set of all contraction sequences on Γ:

$$\underbrace{\min_{\mathcal{S}\in\mathfrak{S}(\Gamma)} \left(\overbrace{\max_{i=1,\ldots,|\mathcal{S}|} \underbrace{(N_i - s_i)}_{\text{local contraction-width of the } i\text{-th contraction}}}^{\text{contraction-width of } \mathcal{S}} \right)}_{\text{contraction-width of } \Gamma}$$

Theorem (L. 2026)

Computing a k-variable Reshetikhin–Turaev knot polynomial on an n-crossing diagram by contracting with contraction-width w is

$$O((\dim V)^w n^{k+2} (\log n)^3).$$

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Lemma (Consequence of [Markov & Shi, 2008])

Let $w(\Gamma)$ and $tw(\Gamma)$ be the contraction-width and tree-width of Γ,

$$w(\Gamma) \leq \Delta(\Gamma) \cdot (tw(\Gamma) + 1) = O(\Delta(\Gamma) \cdot \sqrt{n}),$$

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Corollary

Computing Reshetikhin–Turaev knot polynomials are $e^{O(\sqrt{n})}$.

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Corollary

If the above conjecture is true, the computation of the 2-colored Links–Gould polynomial gives algorithms for the top two items.

The algorithm has been implemented in the in-development version of SnapPy, which computes any RT knot polynomial given its R-matrix and ribbon element.

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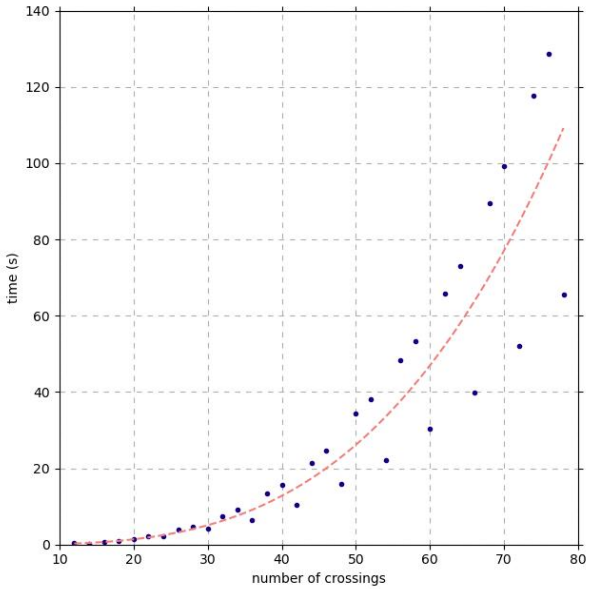
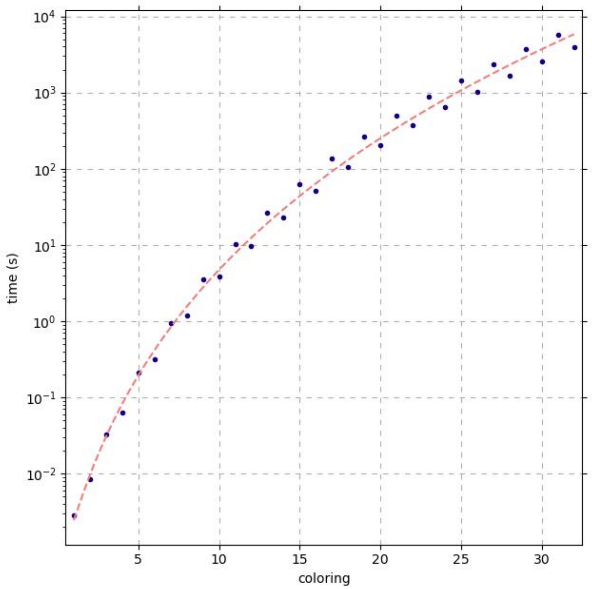


The 2-colored Links–Gould polynomial for $(3, i)$-torus links as i ranges from 6 to 39.



The n-colored Jones polynomial for 11n34 as n ranges from 1 to 32. (plotted in log scale)

Comparison with other implementations on computing the Colored Jones polynomial:

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Coloring	`KnotTheory`	`ColorJones`	Our implementation
1	0.10473	0.512003	
2	5.75993	0.122924	
3	189.518	0.616438	
4	OOT	4.31470	
5		22.3089	
6		97.3698	
7		406.974	
8		1805.60	
9		OOT	

Table: Run time (in seconds) on the knot $11n34$ with a time constraint of $6 \cdot 10^3$ seconds, where OOT denotes that the time exceeded the time constraint.

Coloring	`KnotTheory`	`ColorJones`	Our implementation (`Python`)
1	0.10473	0.512003	0.002832
2	5.75993	0.122924	0.008507
3	189.518	0.616438	0.033057
4	OOT	4.31470	0.062606
5		22.3089	0.208873
6		97.3698	0.319684
7		406.974	0.942618
8		1805.60	1.19045
9		OOT	3.54350
10			3.90816
11			10.2546
12			9.56505
13			26.3211
14			23.0595
15			62.2023
16			50.9203
17			137.517
18			104.946
19			267.455
20			202.984
21			500.720
22			368.573
23			894.917
24			634.879
25			1440.03
26			1023.49
27			2317.58
28			1649.33
29			3704.88
30			2586.03

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2	5.75993	0.122924	0.968588	0.008507
3	189.518	0.616438	1.18240	0.033057
4	OOT	4.31470	1.84138	0.062606
5		22.3089	4.18859	0.208873
6		97.3698	9.60003	0.319684
7		406.974	23.4769	0.942618
8		1805.60	50.6798	1.19045
9		OOT	109.212	3.54350
10			215.333	3.90816
11			419.588	10.2546
12			766.792	9.56505
13			1347.75	26.3211
14			2275.95	23.0595
15			3856.06	62.2023
16			OOT	50.9203
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- n-colored Links–Gould polynomial (harder than $(4n-1)$-colored Jones polynomial).
- The $\mathrm{FK3}$ polynomial (about as hard as 11-colored Jones polynomial).
- The KM_3 polynomial (harder than 15-colored Jones polynomial).