# The complete census of orientable cusped hyperbolic 3-manifolds, up to 10 tetrahedra

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## Table of Contents

1 Introduction

- 2 Extending the census to 10 tetrahedra
- 3 Applications



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4	4 <sub>1</sub>		
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minimal number of tetrahedra in triangulations of manifolds



Tetrahedra	Orientable cusped hyperbolic 3-manifolds				
2	m003	m004			
2	m006	m007	m009	m010	m011
3	m015	m016	m017	m019	

Table: The census of orientable cusped hyperbolic manifolds up to 3 tetrahedra

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#### Burton, 2014

There are precisely 75,956 cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

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- 3 Deduplicate the eligible candidates



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Total	8,373,308	7,468,856	904,452



Deduplication: Group by canonical triangulations computed with SnapPea/SnapPy



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With verified computation, the canonical triangulation produced is a complete invariant of cusped hyperbolic 3-manifolds.



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#### L. 2025

There are precisely 150,730 orientable cusped hyperbolic 3-manifolds triangulable by a minimal of 10 tetrahedra.

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# Corollary

There are only finitely many exceptional Dehn fillings on a 1-cusped hyperbolic 3-manifold.



#### Dunfield, 2019

There are precisely 205,822 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

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Shana Li

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(In progress) Classification of each filling & Verify conjectures





# Conjecture (Dunfield-Friedl-Jackson, 2024)

For any cusped hyperbolic 3-manifold M with  $b_1(M) = 1$ ,

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Hyperbolic homology 3-spheres



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200

The 10-tetrahedra census is available for installation on GitHub.



Figure: QR code to the GitHub repository

Alternatively:

 $shana-y-li.github.io \rightarrow Code \rightarrow snappy\_10\_tets$